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Measuring Power and Satisfaction in Societies with Opinion Leaders: An Axiomatization

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Abstract Opinion leaders are actors who have some power over their followers as they are able to influence their followers' choice of action in certain instances. In van den Brink et al. (2011) we proposed a two-action model for societies with opinion leaders. We introduced a power and a satisfaction score and studied some common properties. In this paper we strengthen two of these properties and present two further properties, which allows us to axiomatize both scores for the case that followers require unanimous action inclinations of their opinion leaders to follow them independently from their own action inclinations.

Keywords: Collective choice, follower, opinion leader, power, satisfaction, axiomatization
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1 Introduction

The concept of opinion leadership plays a considerable role in sociology and marketing. Originally, it goes back to work by the Lazarsfeld group' (see, e.g., Katz and Lazarsfeld 1955, and Lazarsfeld et al. 1968). Following some critique on the original model Troldahl (1966) suggested his so-called *two-cycle flow of communication* model. Roughly speaking, it distinguishes between two phases in the communication process. Phase one is a *flow of information* from a sender (e.g., the mass media) to the members of the society, which is assumed to be a *one-step process*, i.e., the information goes directly to all members of the society. Phase two is a *flow of influence* on beliefs and behavior, which is assumed to be a *two-step process*. In a first step highly self-confident actors with strong opinions who are called *opinion leaders* form their own opinions based on additional information provided by experts, such as academics, while in a subsequent second step their opinion influences their so-called *followers*. These feel attracted by the opinion leaders holding them in high esteem and, under certain conditions, they are prepared to adopt their opinion for their own behavior.[1] This implies that opinion leaders possess some *power over* their followers as they may be able to influence their followers' behavior.[2]

Since Troldahl's contribution the literature on opinion leadership has provided a strong body of knowledge of how and why opinion leaders influence followers choices (see Hoyer and Stockburger-Sauer 2007). Based upon Troldahl's model in van den Brink et al. (2011) we laid the foundations for a theoretical investigation of different *opinion leader-follower structures* and introduced a power and a satisfaction score, which allow us to compare and contrast different structures.

Our basic model considers societies being partitioned into *opinion leaders*, *followers*, and *independent actors*, who have to make a decision on an exogenous proposal. In line with Troldahl's (1966) model we assume that the proposal is distributed among all actors. Following Sinha and Raghavendra (2006) each actor faces a binary choice: by choosing the *yes*-action it can support the proposal in order to obtain a new state of the society, and by choosing the *no*-action it can reject it in order to remain with the status quo. Before making its choice each actor forms its own opinion on the proposal, i.e., without being influenced by any other actor. We call this the actor's *action inclination*.[3] An action inclination contains the information which available state of

[1] Note that opinion leaders provide their followers just with their own opinion on the information distributed in phase one and do not provide them with any additional factual information. In this sense followers consider them as (credible) information sources for the decision regarding their own behavior (see, for instance, Deutschman and Danielson 1960 or, more recently, Solomon et al. 2010).

[2] As a result of this influence, the ability of the followers to determine the outcome of the collective choice, i.e., their *power to* (do something with respect to the outcome of the collective choice), might be affected.

[3] Note that in line with the corresponding generic definitions of an *inclination* and a *preference* in our context we treat both notions as synonyms. We make use of the notion *inclination* as our analysis is closely related to the strand of literature on the measurement

the society an actor prefers to become reality, i.e., whether the outcome of the collective choice should be that the proposal is accepted or rejected. Hence, an actor can either have the action inclination to choose the *yes*- action in order to support the proposal or the *no*-action in order to reject it. This implies that there exists a relationship between an action inclination as defined above and the outcome of the collective choice. In addition to their action inclinations, the actors possess also *constitutional inclinations*, which, in the present context, are related to the organization of the society, or to make use of the words of Vanberg and Buchanan (1988) they are an actor's "preferences over potential alternative 'rules of the game' for the social community or group within which he operates. ... They reflect preferences that would emerge if he were to participate in choosing the constitution, in the broadest sense." In our context they determine (i) whether an actor is an opinion leader, follower, or independent actor, (ii) which opinion leader(s) an actor chooses if it is a follower, and (iii) the procedure for followers to follow their opinion leaders. For the purpose of our analysis we assume that constitutional inclinations are exogenous. This implies that (i) we have a given partition of our society into opinion leaders, followers, and independent actors, and that (ii) our opinion leader-follower relationships are already fixed, i.e., it is given which actors might influence the choice of action of certain other actors by exercising some power over them. Moreover, (iii) we assume that a unanimity requirement applies for followers to follow their opinion leaders. Action inclinations are left 'unspecified'. In line with Vanberg and Buchanan (1988) and Heckathorn (1987), who have introduced the distinction between both types of inclinations, we assume a lexicographical ordering of both types: constitutional inclinations are regarded to be on a 'higher level' than action inclinations as action inclinations have to be formed within the framework given by the constitutional inclinations. As we assume constitutional inclinations to be exogenous from now onwards, when we refer to *action inclinations*, we will just use the term *inclinations*.

In line with the inherent idea of opinion leadership we suppose that via informal discussions of the proposal the inclinations of the opinion leaders are becoming public information prior to the real decision, which implies that each follower is aware of the inclination of its opinion leader(s).[4] Only after these discussions, all actors will secretly (or simultaneously) choose their actions. These correspond with an actor's inclination, if it is an opinion leader or independent actor. If it is a follower with just one opinion leader, it will - independently of its own inclination - adopt the inclination of its opinion leader. If a follower has more than one opinion leader then it will choose an action against its own inclination if, and only if, a certain fraction of its opinion leaders have the same inclination, and this differs from its own inclination. For the purpose of the present paper we consider the specific, but quite usual,

of power which goes back to Hoede and Bakker (1982). What they - and we in van den Brink et al. (2011) - call an *inclination* is called an *action inclination* in the present paper.

[4] While our model allows that the public discussions may change the inclinations of some actors, we suppose that when choosing their action, and after that, inclinations do not alter.

case that this fraction is equal to one, i.e., *unanimity*.[5] Finally, based on the individual choices of all actors, a decision-making mechanism determines the outcome of the collective choice, i.e., whether the proposal is adopted or not. We assume that the collective choice is made by simple majority of the actions chosen by the actors.

As tools for the analysis of different opinion leader-follower structures van den Brink et al. (2011) introduced two scores being based on the notions of *power* and *satisfaction* where the latter contains the former as one component (see van den Brink and Steffen 2012). In the voting power literature *power* is, usually, ascribed to an actor if, given the chosen actions of the others, by changing its own action, the actor changes the outcome. It is said that the actor has a *swing*. Hence, roughly speaking, power in this context is defined as the ability of an actor, i.e., what the actor is able to do (by changing its action) against some resistance of others (represented by those chosen actions of the others, which are not in line with the 'new' action of the actor in question) irrespective of the actual occurrence of this resistance (see van den Brink and Steffen 2008 referring to Braham 2008). While this definition of power is sufficient for many applications, it implies a number of simplifying assumptions (see Morriss 1987/2002:154-156). Among others it assumes (i) that actors choose an action in line with their inclination, and (ii) that each actor's choice of action is not influenced by another actor. The implication of (i) is that for measurement purposes it is sufficient to consider the actions individually chosen by the actors and the resulting outcome, and to ignore the fact that inclinations are usually part of any definition of power defined as an ability. Following Morriss (1987/2002:26), 'abilities are things that we can do *when we want*'. As our opinion leader-follower collective choice situations allow for situations under which both assumptions above are violated, we have to relax both. This implies that for our purposes we have to consider the actors' inclinations for the measurement of power. Hence, in our context we ascribe power to an actor if, given the inclinations of the others, an actor by changing its inclination can enforce a change in the outcome via a change of its action. The power score that is introduced in van den Brink et al. (2011) is based on such power ascriptions and informs us about the power distribution among the members of a society with respect to their ability to affect the state of the society concerning a specific outcome.

According to van den Brink and Steffen (2012) *satisfaction* should be ascribed to an actor if its inclination corresponds with the outcome. This ascription is based upon the generic definition of *satisfaction* being the fulfillment of a desire.[6] As a powerful actor in the sense of the power definition above can

[5] In the current context this assumption appears to be in line with other findings in the literature. For instance, Asch's (1951, 1952, 1956) results imply that when a group takes a unanimous position, people may feel more pressure to conform. A very recent study underpinning this view comes from an experiment conducted by Verhulst and Levitan (2009). They found that participants were more likely to conform to the attitudes expressed by a unanimous group than by a non-unanimous group.

[6] Note that *satisfaction* has to be distinguished from the 'preference free' notion of *success*, which is ascribed if an actor's choice of action corresponds with the outcome. Both notions

itself determine the outcome it desires, power is one feasible source of satisfaction. However, also a powerless actor can gain satisfaction. This satisfaction can result out of different types of *luck* (see van den Brink and Steffen 2012). Hence, we can say that, in our context, satisfaction is an important notion as it is the most extensive notion to characterize the position of an actor in a society (or organization) while the notion of power is of equal importance as it informs us about the fraction of satisfaction that an actor can enforce itself by the choice of its own action.[7] The satisfaction score introduced in van den Brink et al. (2011) and which is applied in this paper is based on the satisfaction ascription defined above and, hence, tells us to which degree members of the society can be expected to end up with an outcome that they desire. [8]

The paper is structured as follows. In Sect. 2 we provide a formal representation of our model and our power and satisfaction scores. In Sect. 3, which contains the main contribution of the present paper, we show that, in the present context, our power and satisfaction scores satisfy even stronger opinion leader properties than those studied in van den Brink et al. (2011). Moreover, we introduce two different normalizations and provide full axiomatizations of both scores, which differ in the normalization only. Finally, in Sect. 4 we wind-up the paper with a concluding remark.

2 Measuring power and satisfaction in opinion leader-follower collective choice situations

2.1 The model

We represent the structure of our society and the 'opinion leader-follower' relations by a *bipartite directed graph* (or *bipartite digraph*) (N, D) with a finite set of nodes $N = \{1, \ldots, n\}$ representing the actors, and $D \subset N \times N$ a

coincide for the canonical set-up in the voting power literature and, hence, are often used as synonyms. However, for the model in the present paper this is not the case. See van den Brink and Steffen (2012) for a more detailed and critical discussion of the relationship between both notions.

[7] Consider, for example, collective decision making in the EU. In several countries there is a growing opposition against 'giving away sovereignty to Brussels'. This opposition clearly focuses on power and ignores satisfaction. However, if a country aimed to increase its satisfaction with the decisions made by the EU decision-making bodies, it might be useful to accept a transfer of their current EU decision-making rights and, hence, their power, to the EU. For example, suppose that the inclinations of The Netherlands and Germany coincide. Then, if the Germans were acting as de facto dictators in the EU decision-making bodies, The Netherlands having no longer any decision-making rights (and, hence, no power) would always be satisfied even they could never be successful. Instead, if the decision-making rights and power were more spread over the countries, one could expect that The Netherlands had some power and would have been successful in some instances, but would have been less satisfied. Hence, for the 'optimal' design of EU collective decision-making mechanisms one needs to know the importance of satisfaction and its components such as power and success for the member countries.

[8] Note that this outcome needs not be in line with the action an actor has chosen. This would have been the case if we had applied the notion of *success* instead of *satisfaction*.

binary relation on N expressing that each actor is either an opinion leader, a follower, or an independent actor. Since we take the set of actors N fixed, we can represent a digraph (N, D) by its binary relation D. Let $S_D(k) = \{j \in N : (k, j) \in D\}$ be the set of successors of actor $k \in N$ in digraph D, and let $P_D(k) = \{j \in N : (j, k) \in D\}$ be the set of k's predecessors in D. As we assume that each actor is either an opinion leader, follower or independent actor, we consider digraphs D such that:

$$|S_D(k)| \cdot |P_D(k)| = 0 \text{ for each } k \in N, \tag{2.1}$$

where $|X|$ denotes the cardinality of set X. Let $OL(D) = \{k \in N : S_D(k) \neq \emptyset\}$ be the set of opinion leaders, $FOL(D) = \{k \in N : P_D(k) \neq \emptyset\}$ be the set of followers, and $IND(D) = N \setminus (OL(D) \cup FOL(D))$ be the set of independent actors in D. Since, by assumption (2.1) we have that $OL(D) \cap FOL(D) = \emptyset$, the sets $OL(D)$, $FOL(D)$, and $IND(D)$ form a partition of the set N. We denote the collection of all bipartite digraphs on N by $\mathcal{D}^N$.

As explained in Sect. 1 we assume, that after the distribution of an exogenous proposal each actor will form its own inclination. The chosen inclinations are represented by the inclination vector $I = (I_1, ..., I_n) \in \{0, 1\}^n$. This is a vector which k^{th} component, I_k, is 1 if actor k has the inclination to choose the *yes*-action in order to support the proposal, and 0 if it has the inclination to choose the *no*-action in order to reject it. Now, we can define an *opinion leader-follower collective choice situation* as a pair (I, D) with $I \in \{0, 1\}^n$ and $D \in \mathcal{D}^N$.

Next, let $V = V(I, D) \in \{0, 1\}^n$ denote the choice vector, that is a vector which k^{th} component, V_k, is 1 if actor k has chosen the *yes*-action, and 0 if k has chosen the *no*-action. Thus, the choice vector $V = V(I, D) \in \{0, 1\}^n$ is given by:

$$V_k = I_k \text{ if } k \in OL(D) \cup IND(D),$$

and for $k \in FOL(D)$:

$$V_k = \begin{cases} x & \text{if } I_j = x \text{ for all } j \in P_D(k) \\ I_k & \text{otherwise,} \end{cases} \tag{2.2}$$

where $x \in \{0, 1\}$.

After all actors have chosen their actions, an outcome is resulting according to the decision-making mechanism in use. This is given by the *collective decision function* $C\colon \{0, 1\}^n \times \mathcal{D}^N \to \{0, 1\}$, which assigns an outcome to every pair $(I, D) \in \{0, 1\}^n \times \mathcal{D}^N$, that is the value 0 if the collective decision is to *reject*, and the value 1 if the collective decision is to *accept* the proposal. We define the collective decision function by simple majority voting. Let for an action $x \in \{0, 1\}$ and choice vector $V = V(I, D) \in \{0, 1\}^n$ the number of actors choosing the action x be denoted by $n_x(V(I, D)) = |\{k \in N : V_k = x\}|$. Restricting our analysis to situations in which the number of actors is odd, the collective decision function is defined, for each (I, D), as follows:

$$C(I, D) = \begin{cases} 1 & \text{if } n_1(V(I, D)) > n_0(V(I, D)) \\ 0 & \text{if } n_0(V(I, D)) > n_1(V(I, D)). \end{cases} \tag{2.3}$$

2.2 Measuring power and satisfaction

In general, a score for bipartite digraphs representing opinion leader-follower collective choice situations is a function $f\colon \mathcal{D}^N \to \mathbb{R}^n$, which assigns an n-dimensional real vector to every bipartite digraph on N. Here we consider the power and satisfaction scores introduced by van den Brink et al. (2011).

As explained in Sect. 1 we ascribe power to an actor if the actor, by changing its inclination, is able to alter the outcome, i.e., if the actor has a *swing*, and we measure the power of the actor in an opinion leader-follower collective choice situation by the actor's likelihood to be powerful. Formally, an actor $k \in N$ has a *swing* in (I, D) according to collective decision function C if $C(I, D) \neq C(I', D)$ with $I'_k \neq I_k$ and $I'_j = I_j$ for all $j \in N \setminus \{k\}$. In order to ascribe power to actor k we define a power score of actor k under a given inclination vector:

$$\overline{POW}_k(I, D) = \begin{cases} 1 & \text{if } k \text{ has a swing in } (I, D) \\ 0 & \text{otherwise.} \end{cases}$$

Then, the *power* score $POW\colon \mathcal{D}^N \to \mathbb{R}^n$ is given by:

$$POW_k(D) = \sum_{I \in \{0,1\}^n} \overline{POW}_k(I, D) \quad \text{for each } k \in N. \tag{2.4}$$

As explained in Sect. 1 we ascribe satisfaction to an actor if the actor's inclination corresponds with the outcome and we measure the satisfaction of the actor in an opinion leader-follower collective choice situation by the actor's likelihood to be satisfied. Formally, in order to ascribe satisfaction to an actor we define a satisfaction score of an actor under a given inclination vector, i.e., for each $(I, D) \in \{0, 1\}^n \times \mathcal{D}^N$ and $k \in N$:

$$\overline{SAT}_k(I, D) = \begin{cases} 1 & \text{if } C(I, D) = I_k \\ 0 & \text{otherwise.} \end{cases}$$

Then, the *satisfaction* score $SAT\colon \mathcal{D}^N \to \mathbb{R}^n$ is given by:

$$SAT_k(D) = \sum_{I \in \{0,1\}^n} \overline{SAT}_k(I, D) \quad \text{for each } k \in N. \tag{2.5}$$

3 Axiomatizations

In van den Brink et al. (2011) it is shown that both scores satisfy the following four properties. Let us denote a generic score by $f\colon \mathcal{D}^N \to \mathbb{R}^n$.

Symmetry If $S_D(k) = S_D(j)$ and $P_D(k) = P_D(j)$, then $f_k(D) = f_j(D)$.
Dictator property If $D \in \mathcal{D}^N$ and $h \in N$ is such that $S_D(h) = N \setminus \{h\}$, then $f_h(D) = 2^n$.

Dictated independence If $D, D' \in \mathcal{D}^N$ and $k \in N$ are such that $|P_D(k)| = |P_{D'}(k)| = 1$, then $f_k(D) = f_k(D')$.

Opposite gain property Let $D, D' \in \mathcal{D}^N$, $j \in IND(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') - f_h(D) = f_j(D) - f_j(D')$.

Symmetry reflects a well-known principle stating that the value of a score for actors with a symmetric position in the bipartite digraph is the same. The dictator property states that, if there is a dictator (being a unique opinion leader who is followed by all other actors), then the score of the dictator is equal to the total number of possible inclination vectors. For the power and satisfaction scores this reflects that a dictator has the power to change the outcome by changing its own inclination and, if the dictator votes according to its inclination, then the outcome will be the inclination of the dictator. Dictated independence states that the score of a follower with one opinion leader does not change as long as this follower is dictated by a sole opinion leader, and reflects the idea that a follower who has only one opinion leader will always follow this opinion leader. The opposite gain property states that, if an actor becomes a sole opinion leader of another actor who was previously independent, then the sum of the scores of these two actors does not change. In other words, if the opinion leader gains, then this goes fully at the cost of the follower.[9]

Next, we strengthen two axioms that were introduced in van den Brink et al. (2011). First, *horizontal neutrality* states that, if a follower gets one more opinion leader, then the sum of scores of the 'new' and an 'old' opinion leader does not change. In other words, the change for the new opinion leader is opposite but in absolute value equal to the change for an 'old' opinion leader.[10]

Second, the *equal gain property* states that, if a follower gets one more opinion leader, then the changes in scores of this follower and of its new opinion leader are the same.[11]

Horizontal neutrality Let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $g \in P_D(j)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') - f_h(D) = f_g(D) - f_g(D')$.

Equal gain property Let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') - f_h(D) = f_j(D') - f_j(D)$.

[9] This principle is similar to several collusion neutrality properties introduced in the context of cooperative TU-games by e.g., Lehrer (1988), Haller (1994) and Malawski (2004), and applied to the so-called games with a hierarchical permission structure in van den Brink (2010).

[10] This is a stronger version of *power neutrality for two opinion leaders*.

[11] This is a stronger version of the *equal absolute change property*. It is similar to Myerson's (1977) *fairness* for cooperative communication graph games, where it states that deleting a communication link between two players changes their individual payoffs by the same amount. In van den Brink (1997) this type of equal gain/loss property is stated in terms of the above mentioned games with a hierarchical permission structure.

Besides showing that the power and satisfaction scores satisfy these last two stronger axioms, the main contribution of this paper is to show that adding a further axiom which specifies what is the total sum of the scores over all actors (i.e., a normalization), characterizes the power and satisfaction scores. Obviously, since power and satisfaction are related but different concepts (see, e.g., Dowding 1996 and van den Brink and Steffen 2012) each will satisfy a different normalization. As normalization of power we take that the sum of all scores is equal to the total number of individual swings, i.e., for each inclination vector we count how many actors have a swing, and we add all these swings over all inclination vectors. As normalization of satisfaction we take that the sum of all scores is equal to the total number of individual satisfaction ascriptions, i.e., for each inclination vector we count how many actors have an inclination that corresponds with the outcome, and we add all these satisfaction ascriptions over all inclination vectors.

Power normalization For every $D \in \mathcal{D}^N$ it holds that $\sum_{k \in N} f_k(D) = \sum_{I \in \{0,1\}^n} |\{k \in N : k \text{ has a swing in } (I, D)\}|$.

Satisfaction normalization For every $D \in \mathcal{D}^N$ it holds that $\sum_{k \in N} f_k(D) = \sum_{I \in \{0,1\}^n} |\{k \in N : I_k = C(I, D)\}|$.

Next, we state the main result of this paper.

Theorem 1 *(i) Let the choice vector V be defined by (2.2). A score $f : \mathcal{D}^N \to \mathbb{R}^n$ is the power score $POW : \mathcal{D}^N \to \mathbb{R}^n$ if, and only if, it satisfies symmetry, the dictator property, dictated independence, the opposite gain property, horizontal neutrality, the equal gain property, and power normalization.*
(ii) Let the choice vector V be defined by (2.2). A score $f : \mathcal{D}^N \to \mathbb{R}^n$ is the satisfaction score $SAT : \mathcal{D}^N \to \mathbb{R}^n$ if, and only if, it satisfies symmetry, the dictator property, dictated independence, the opposite gain property, horizontal neutrality, the equal gain property, and satisfaction normalization.

Proof
(i) *POW* satisfying symmetry, the dictator property, dictated independence, and the opposite gain property is shown in van den Brink et al. (2011). It is obvious that *POW* satisfies power normalization.

To show the equal gain property and horizontal neutrality[12], let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Note that $\overline{POW}_k(I, D) = 1$ implies that (i) $\overline{POW}_k(I, D') = 1$ for $k \in \{h, j\}$, and (ii) $\overline{POW}_k(I, D) = 0$ implies that $\overline{POW}_k(I, D') = 0$ for all $k \in P_D(j)$. Since $[\overline{POW}_h(I, D) = 0$ and $\overline{POW}_h(I, D') = 1]$ if, and only if, $[I_j = I_h \neq I_k$ for all $k \in P_D(j)$ and $C(I', D) \neq C(I, D)$ for $I'_j = I'_h \neq I_h$ and $I'_k = I_k$ for all $k \in N \setminus \{h, j\}]$ if, and only if, $[\overline{POW}_j(I, D) = 0$ and $\overline{POW}_j(I, D') = 1]$ if, and only if, $[\overline{POW}_k(I, D) = 1$ and $\overline{POW}_k(I, D') = 0$ for all $k \in P_D(j)]$, *POW* satisfies the equal gain property and horizontal neutrality.

[12] In van den Brink et al. (2011) it is already shown that horizontal neutrality holds, if the follower has exactly one opinion leader.

To prove uniqueness, suppose that the score $f: \mathcal{D}^N \to \mathbb{R}^n$ satisfies the seven axioms, and let $D \in \mathcal{D}^N$. We prove that f must be equal to *POW* in several steps. We first prove uniqueness in case there is at most one actor that is an opinion leader by induction on its number of followers.

First, suppose that the opinion leader is a dictator, i.e., there is an $h \in N$ such that $S_D(h) = N \setminus \{h\}$. Then the dictator property implies that $f_h(D) = 2^n$. Since each actor $k \in N \setminus \{h\}$ has no swing, power normalization implies that $\sum_{k \in N} f_k(D) = 2^n$, and, thus, $\sum_{k \in N \setminus \{h\}} f_k(D) = 2^n - 2^n = 0$. Symmetry then implies that $f_k(D) = 0$ is determined for all $k \in N \setminus \{h\}$.

Proceeding by induction, suppose that $f(\widehat{D})$ is uniquely determined whenever $|S_{\widehat{D}}(h)| > |S_D(h)|$. Take a $j \in N \setminus (\{h\} \cup S_D(h))$. Note that j is an independent actor since h is the only actor with successors. Consider $D' \in \mathcal{D}^N$ given by $D' = D \cup \{(h,j)\}$. Dictated independence and the induction hypothesis imply that $f_k(D)$ is uniquely determined for all $k \in S_D(h)$. Symmetry implies that there is a constant $c \in \mathbb{R}$ such that:

$$f_k(D) = c \text{ for all } k \in N \setminus (\{h\} \cup S_D(h)). \tag{3.6}$$

The opposite gain property implies that:

$$f_h(D') - f_h(D) = f_j(D) - f_j(D'), \tag{3.7}$$

where $f_h(D')$ and $f_j(D')$ are given by the induction hypothesis. Then, with power normalization, (3.6) and (3.7) yield $(n - 1 - |S_D(h)|) + 1 + 1 = n - |S_D(h)| + 1$ linearly independent equations with the $n - |S_D(h)| + 1$ unknowns, c and $f_i(D),\ k \in N \setminus S_D(h)$. Thus, $f(D)$ is uniquely determined.

Next, we prove that $f(D)$ is uniquely determined for all $D \in \mathcal{D}^N$ by induction on $|D|$.
From above it follows that $f(D)$ is uniquely determined if $D = \emptyset$.[13]
Proceeding by induction, assume that $f(\widehat{D})$ is uniquely determined whenever $|\widehat{D}| < |D|$.
We distinguish the following cases with respect to actor $k \in N$ (of which at least one must occur):

1. If $|P_D(k)| = 1$, then dictated independence and the case with a dictator considered before imply that $f_k(D) = 0$ is uniquely determined.
2. If there is an $j \in S_D(k)$ with $|P_D(j)| = 1$, then the opposite gain property implies that

$$f_k(D) + f_j(D) = f_k(D \setminus \{(k,j)\}) + f_j(D \setminus \{(k,j)\}). \tag{3.8}$$

Since actor j is as in case 1, we determined $f_j(D)$. With the induction hypothesis $f_k(D \setminus \{(k,j)\})$ and $f_j(D \setminus \{(k,j)\})$ are determined. Thus, with (3.8), $f_k(D) = f_k(D \setminus \{(k,j)\}) + f_j(D \setminus \{(k,j)\}) - f_j(D)$ is uniquely determined.

[13] Note that this also follows from symmetry and power normalization.

3. If there is an $j \in S_D(k)$ with $|P_D(j)| \geq 2$, then take $h \in P_D(j) \setminus \{k\}$. The equal gain property implies that
$$f_k(D) - f_k(D \setminus \{(k,j)\}) = f_j(D) - f_j(D \setminus \{(k,j)\}) \tag{3.9}$$
and
$$f_h(D) - f_h(D \setminus \{(h,j)\}) = f_j(D) - f_j(D \setminus \{(h,j)\}). \tag{3.10}$$
Horizontal neutrality implies that:
$$f_k(D) - f_k(D \setminus \{(k,j)\}) = f_h(D \setminus \{(k,j)\}) - f_h(D). \tag{3.11}$$
With the induction hypothesis the values in graphs $D \setminus \{(k,j)\}$ and $D \setminus \{(h,j)\}$ are uniquely determined. Thus, with the three linearly independent equations (3.9), (3.10) and (3.11), the scores $f_k(D)$, $f_j(D)$ and $f_h(D)$ are uniquely determined.
4. If $|P_D(k)| \geq 2$, then $f_k(D)$ is uniquely determined as in the previous case (with the roles for k and j reversed).
5. Finally, symmetry implies that there is a constant $c \in \mathbb{R}$, such that $f_k(D) = c$ for all $k \in IND(D)$. Since above we determined all $f_j(D)$ for $j \in OL(D) \cup FOL(D)$, power normalization determines c.

Thus, all $f_k(D)$, $k \in N$, are uniquely determined.

(ii) *SAT* satisfying symmetry, the dictator property, dictated independence, and the opposite gain property is shown in van den Brink et al. (2011). It is obvious that *SAT* satisfies satisfaction normalization.

To show the equal gain property and horizontal neutrality, let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h,j)\}$. If $C(I,D) \neq C(I,D')$, then it must hold that actor j initially had to vote against its inclination and now can vote according to its inclination, because its new opinion leader h has the same inclination. Moreover, all 'other' opinion leaders of j must have the opposite inclination. Thus, for $g \in P_D(j)$ we have $C(I,D) = I_g \neq I_j = I_h$ and $C(I,D') = I_j = I_h \neq I_g$. So, $\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D) = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_g(I,D) - \overline{SAT}_g(I,D') = 1$.

Obviously, if $C(I,D) = C(I,D')$, then $\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D) = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_g(I,D) - \overline{SAT}_g(I,D') = 0$. Thus, with (2.5) we have $SAT_j(D) - SAT_j(D') = SAT_h(D) - SAT_h(D') = SAT_g(D') - SAT_g(D)$, showing that *SAT* satisfies the equal gain property and horizontal neutrality.

Uniqueness follows similar as the uniqueness proof of part (i), but using the alternative satisfaction normalization. □

Note that Theorem 1 characterizes the power score *POW* and the satisfaction score *SAT* by the same axioms except the normalization axiom. Thus, we have two comparable axiomatizations which exactly illustrate the difference between power and satisfaction, which lies in the normalization being applied.[14] This expresses the basic difference in measuring power and satisfaction.

[14] A similar difference is shown by van den Brink and Gilles (2000) for the outdegree measure and the β-measure as scoring methods for directed graphs, highlighting that a normalization is not always so innocent as it might appear.

4 Concluding remark

Having demonstrated that our power and satisfaction scores differ only in the normalization that is used, we would like to wind-up this paper with a remark on another difference between both scores. They also differ in how they treat situations where a dictator exists. Power normalization and the dictator property imply that the power score *POW* satisfies the property, that the power score of actors who are subordinates to a dictator is equal to zero.[15] Instead, satisfaction normalization and the dictator property imply, that the satisfaction score *SAT* satisfies the property that in case there is a dictator, the satisfaction score of a dictated actor is half of the satisfaction score of the dictator. This expresses another difference between measuring satisfaction and power. If a dictator exists, then the other actors cannot influence the outcome of the voting process since they have to follow the dictator, and therefore their power is equal to zero. However, also an actor that is subordinate to a dictator might end up with an outcome that corresponds with its inclination. Thus, ex ante a subordinate of a dictator will have its inclination coinciding with that of the dictator in half of the cases. Since a dictator always dictates the outcome, we arrive at a satisfaction score of the subordinate that is half of the satisfaction score of the dictator.

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[15] The power score *POW* even satisfies a stronger property, which states that if a follower has a unique opinion leader, then its power is equal to zero, i.e., if $D \in \mathcal{D}^N$, $j \in \mathit{FOL}(D)$ and $h \in \mathit{OL}(D)$ such that $P_D(j) = \{h\}$, then $f_j(D) = 0$.

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